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PT-symmetric, quasi-exactly solvable matrix Hamiltonians

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Abstract

Matrix quasi-exactly solvable operators are considered and new conditions are determined to test whether a matrix differential operator possesses one or several finite-dimensional invariant vector spaces. New examples of (2×2)-matrix quasi-exactly solvable operators are constructed with the emphasis set on PT-symmetric Hamiltonians.

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1. Introduction

Several problems in quantum physics lead to the mathematical challenge of determining the spectrum of a linear operator defined on an appropriate space of functions. Unfortunately, the relevant spectrum can be computed exactly (i.e. by means of algebraic methods) only in a few particular cases. In the last few years, an intermediate class of operators has been discovered [1–4]: the quasi-exactly solvable (QES) operators, for which a *finite* part of the spectrum can be computed algebraically.

Since this paper is largely devoted to QES operators, we recall the definition which is used in the recent literature [5]. Let a one-dimensional operator of the form $H = -\partial_x^2 + V(x)$ be essentially self-adjoint on an Hilbert space $\mathcal{H}$; H is said *quasi-exactly solvable* if it leaves invariant one (or more) non-trivial finite-dimensional subspace $\mathcal{M}$ of $\mathcal{H}$, in other words if

$$\mathcal{M} = \mathrm{span}\{\phi_1, \ldots, \phi_n, \phi_j \in \mathcal{H}\} : H\phi_j \in \mathcal{M}, \qquad j = 1, \ldots, n.$$

This definition can be naturally extended to more general contexts, e.g. to operators depending on several variables or to matrix differential operators.

Several QES operators are equivalent to operators preserving a ring of polynomials of given degree in a variable z. By equivalent, it is understood that a change of function (so-called gauge transformation, depending on an invertible function $\mu(x)$) and/or a change of variable $y = y(x)$ have to be performed on the initial Schrödinger operator $H(x)$ in order to reveal its invariant vector space in terms of a space of polynomials. These QES operators are closely related to the representations of the SL(2,ℝ) Lie algebra [1]; however, several QES operators have been constructed beyond the framework of the Lie algebra [5].

In the case of coupled equations, where the operators appear in the form of a matrix whose components are differential operators, the construction of the gauge transformation setting the operator in a form which manifestly preserves a vector space where components are polynomials, turns out to be more tricky (see, e.g., [6, 7]).

In the second section of this paper, we establish a set of algebraic conditions to test whether $(n \times n)$-matrix-valued operators of a certain type preserve a vector space of n-uple of polynomials with component of definite degrees. We work with 2×2 matrices but the method can be extended to higher dimensions. This new method was tested on all QES-known matrix equations. In the other sections, we take advantage of these conditions and construct several new families of QES systems where the emphasis is set on PT-symmetric invariance. This original issue for the mathematical framework of quantum mechanics was proposed in [8] and developed in several subsequent papers, but to our knowledge it has not been studied in the context of coupled systems of Schrödinger equations.

In section 3, we propose several matrix extensions of the Razhavi operator. Scalar Razhavi-types of potentials were considered recently to produce examples of PT-invariant, non-Hermitian potentials with real eigenvalues [9, 10]. Here, we develop matrix extensions of them both with trigonometric and hyperbolic potentials.

In section 4, we obtain a matrix generalization of the QES example of the PT-symmetric Hamiltonian with an anharmonic potential of degree four [8] and reconsidered recently [11]. Finally, in section 5, we show how the problem of section 4 can be transformed into a system of recurrence equations in the spirit of [12].

2. Matrix QES operator

In this section, we propose a test to check whether a family of (2×2)-matrix differential operator, H, preserves a vector space whose components are polynomials of suitable degrees in x. We consider the case where the components of H are combinations of the derivatives $\mathrm{d}^n/\mathrm{d}x^n$ with polynomial coefficients. More precisely, we study the family of operators H which can be decomposed according to

$$H = H_1 + H_0 + H_{-1} + H_{-2}, \qquad H_s = \begin{pmatrix} \hat{A}_s & b_s x^{\delta+1-s} \\ c_s x^{\delta'+1-s} & \hat{B}_s \end{pmatrix}. \tag{1}$$

Here $\hat{A}_s$, $\hat{B}_s$ represent homogeneous differential operators of degree s, i.e. they transform the monomial x^m into a monomial proportional to x^{m+s} for $m \in \mathbb{N}$; b_s, c_s, are arbitrary constants and δ, δ' are integers. It is also understood that the off-diagonal components of H do not contain negative powers of x.

Now, we try to obtain the conditions of the different constants entering in H, such that this operator possesses an invariant subspace of polynomials of the form

$$\mathcal{V} = \text{span}\left\{ \begin{pmatrix} p_n \\ q_m \end{pmatrix}, n, m \in \mathbb{N} \right\}, \tag{2}$$

where p_n, q_m denote polynomials of degree n, m in the variable x. Requiring, for physical motivations, H to contain effectively derivatives up to second order, we have a first necessary condition:

Condition 1. $\mathcal{V}$ can be an invariant vector space of H only if

$$\delta + \delta' = 2, \qquad n - m = \delta - 1 \qquad \text{for} \quad \delta = 0, 1, 2. \tag{3}$$

This condition can be demonstrated just by a calculation. If it does not hold, then the operator H is trivial, or not of a form suitable for quantum mechanics. However the condition (3) is NOT sufficient.

In order to obtain necessary and sufficient conditions, it is useful to define some notations. Let us consider a generic vector in $\mathcal{V}$

$$\psi = \begin{pmatrix} \alpha_0 x^n \\ \beta_0 x^{n-\delta+1} \end{pmatrix} + \begin{pmatrix} \alpha_1 x^{n-1} \\ \beta_1 x^{n-\delta} \end{pmatrix} + \cdots, \tag{4}$$

where α_j, β_j are arbitrary complex parameters and let the operator H acts on this vector. The components of the vector $H\psi$ are then polynomials in x whose components are *linear* in the constants α_j, β_j. As a consequence the vector $H\psi$ can be decomposed *uniquely* according to

$$\begin{aligned} H\psi = {} & \mathrm{diag}(x^{n+1}, x^{n-\delta+2}) M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \\ & + \left(\mathrm{diag}(x^n, x^{n-\delta+1}) \tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + \mathrm{diag}(x^n, x^{n-\delta+1}) M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \right) \\ & + \text{terms of lower degrees in } x. \end{aligned} \tag{5}$$

This defines in particular the constant 2×2 matrices M_1, $\tilde{M}_1$ and M_0. They can be computed explicitly after a straightforward calculation once the explicit form of H is chosen. The following result is easily obtained from (5).

Condition 2. The necessary and sufficient conditions for H to admit $\mathcal{V}$ as invariant vector space are

$$\text{(i)} \quad M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

$$\text{(ii)} \quad \tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \propto \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}, \tag{7}$$

where the second condition has to be fulfilled irrespectively of the values α_1, β_1.

The condition (i) implies $\det M_1 = 0$ and the vector $(\alpha_0, \beta_0)^t$ to be a zero-eigenvalue eigenvector of M_1. This fixes the relative coefficient of the terms of highest degree in $\mathcal{V}$ (see equation (2)). The condition (ii) is equivalent to the following conditions:

$$\text{(ii}'\text{)} \quad M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \Lambda \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}, \qquad \text{(ii}''\text{)} \quad \tilde{M}_1^t \left(-\beta_0 \alpha_0 \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{8}$$

where M^t means the transpose matrix of M, (ii$'$) and (ii$''$) are in general easier to implement than (ii).

In summary, conditions (i), (ii$'$), (ii$''$) allow to construct in a systematic way the invariant vector spaces for operators of the form (1), and then define criteria for these operator to be QES. The conditions on the coupling constants obtained, e.g., in [7, 14, 15] can be rediscovered along these lines, emerging now in terms of elementary manipulations on the matrices M_1, $\tilde{M}_1$, M_0.

In order to illustrate this method, we reconstruct the invariant vector space of the QES Hamiltonian [14, 15]

$$H(y) = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} \mathbb{1}_2 + M_6(y), \tag{9}$$

the potential $M_6(y)$ is a (2×2) Hermitian matrix of the form

$$\begin{aligned} M_6(y) = {} & \left\{ 4p_2^2 y^6 + 8p_1 p_2 y^4 + \left(4p_1^2 - 8mp_2 + 2(1-2\epsilon)p_2 \right) y^2 \right\} \mathbb{1}_2 \\ & + (8p_2 y^2 + 4p_1)\sigma_3 - 8mp_2 \kappa_0 \sigma_1, \end{aligned} \tag{10}$$

where p_1, p_2, ϵ are constants, m is an integer and σ_a, $a = 1, 2, 3$, denote the Pauli matrices.

The 'gauge transformation' of $H(y)$ with a factor

$$\phi(y) = y^{\epsilon} \exp - \left\{ \frac{p_2}{2} y^4 + p_1 y^2 \right\}, \qquad \epsilon = 0, 1 \tag{11}$$

and the change of variable $x = y^2$ leads to an operator $\tilde{H}(x)$

$$\tilde{H}(x) = \phi^{-1}(y) H(y) \phi(y)|_{y=\sqrt{x}}, \tag{12}$$

which reveal the invariant subspace of H. Defining $J_+(m) \equiv x^2 d_x - mx$ and setting for simplicity $\epsilon = 0$, $p_1 = 0$, we find

$$\tilde{H}(x) = \left(-4x \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2 \frac{\mathrm{d}}{\mathrm{d}x} \right) 1\!\!1_2 + 8 p_2 \begin{pmatrix} J_+(m-2) & 0 \\ 0 & J_+(m) \end{pmatrix} - 8 m p_2 \kappa_0 \sigma_1. \tag{13}$$

This operator can be decomposed along the lines of equation (1),

$$\tilde{H}(x) = H_1 + H_0 + H_{-1}, \tag{14}$$

with

$$\begin{aligned} H_1 &= 8 p_2 \begin{pmatrix} J_+(m-2) & -m\kappa_0 \\ 0 & J_+(m) \end{pmatrix}, \qquad H_0 = 0, \\ H_{-1} &= \left(-4x \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2 \frac{\mathrm{d}}{\mathrm{d}x} \right) 1\!\!1_2 - 8 m p_2 \kappa_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

In this case, $(H_1)_{12}$ is a constant (i.e. $\delta = 0$), while $(H_1)_{21} = 0$, the matrix operator H_0 is zero. The invariant vector space is of the form

$$\psi = \begin{pmatrix} \alpha_0 x^{m-1} + \alpha_1 x^{m-2} + \cdots \\ \beta_0 x^m + \beta_1 x^{m-1} + \cdots \end{pmatrix}. \tag{16}$$

The determinant of the matrix M_1 is trivially zero and the condition (i) implies $\frac{\alpha_0}{\beta_0} = m\kappa_0$. The first conditions (ii′) are trivial since $M_0 = 0$ (as a consequence of $H_0 = 0$). Finally, the second condition (ii′) can be easily checked:

$$\tilde{M}_1^t \begin{pmatrix} -\beta_0 \\ \alpha_0 \end{pmatrix} = -8 p_2 \begin{pmatrix} 0 & 0 \\ m\kappa_0 & 1 \end{pmatrix} \begin{pmatrix} -\beta_0 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

In the following section, we will present several examples of QES matrix operators based on extensions of the scalar Razavi potential.

3. PT-invariant non-Hermitian matrix Hamiltonian

In [9, 10], PT-invariant models based on the scalar Razhavi potential are analyzed with the emphasis set on the reality properties of the spectrum. This can be done partly in an analytical way because the potentials considered are QES. The authors considerd both hyperbolic and trigonometric cases invoking an anti-isospectral transformation [13] to relate the spectra of both types. Here, we will consider matrix extensions of these equations and see that several form of the non-diagonal elements H_{12} and H_{21} can lead to QES operators. We will first consider periodic potentials, formulated in terms of trigonometric functions. The cases of potentials involving hyperbolic functions will be presented afterwards.

3.1. Trigonometric case

From the unidimensional potential studied in [9], we will build a family PT-invariant matrix Hamiltonian and use the technique developed in the previous section to check its quasi-exactly solvability. We start from a general Hamiltonian of the form

$$H = \begin{pmatrix} -\frac{d^2}{dx^2} + (\rho\cos 2x - iM)^2 + A & H_{12} \\ H_{21} & -\frac{d^2}{dx^2} + (\rho\cos 2x - i\tilde{M})^2 + \tilde{A} \end{pmatrix}, \quad (18)$$

where ρ is a free real parameter and $A, \tilde{A}, M, \tilde{M}$ are constants to be specified. There are several forms of H_{12}, H_{21} which lead to QES operators. One can assume $\tilde{M} > M$ without loosing generality. The general properties of the diagonal component of H and of trigonometric functions will reveal that QES operators can be constructed by choosing H_{12} according to one of the following form:

$$\begin{gathered} H_{12} = C\cos 2x + D \qquad \text{or} \qquad H_{12} = C\cos x \qquad \text{or} \\ H_{12} = C\sin x \qquad \text{or} \qquad H_{12} = C\cos x\sin x \end{gathered} \quad (19)$$

and similar forms, respectively, for H_{21} with, however, *a priori* independent coupling constants for C and D.

In order to reveal the algebraic properties of this family of operators, it is convenient to perform a first gauge transformation according to

$$\begin{aligned} \tilde{H} &= e^{-\theta\cos 2x} \begin{pmatrix} z^{-\epsilon}(1-z)^{-\phi} & 0 \\ 0 & z^{-\tilde{\epsilon}}(1-z)^{-\tilde{\phi}} \end{pmatrix} H\, e^{\theta\cos 2x} \begin{pmatrix} z^{\epsilon}(1-z)^{\phi} & 0 \\ 0 & z^{\tilde{\epsilon}}(1-z)^{\tilde{\phi}} \end{pmatrix}, \\ &= \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{pmatrix}, \end{aligned} \quad (20)$$

where $z = (\cos 2x + 1)/2$. Further choosing the parameter θ, according to $\theta = i\frac{\rho}{2}$ the components of $\tilde{H}$ are obtained:

$$\begin{aligned} \tilde{H}_{11} &= -4z(1-z)\frac{d^2}{dz^2} + 2(2z - 1 - 4(1-z)\epsilon + 4\phi z)\frac{d}{dz} + \rho^2 - M^2 + 8\phi\epsilon + 2\epsilon + 2\phi + A \\ &\qquad - 8i\rho\left(z(1-z)\frac{d}{dz} + \epsilon(1-z) - \phi z + \frac{M-1}{4}(2z-1)\right) \\ \tilde{H}_{12} &= z^{\tilde{\epsilon}-\epsilon}(1-z)^{\tilde{\phi}-\phi}H_{12}, \\ \tilde{H}_{21} &= z^{\epsilon-\tilde{\epsilon}}(1-z)^{\phi-\tilde{\phi}}H_{21}, \\ \tilde{H}_{22} &= -4z(1-z)\frac{d^2}{dz^2} + 2(2z - 1 - 4(1-z)\tilde{\epsilon} + 4\tilde{\phi} z)\frac{d}{dz} + \rho^2 - M^2 + 8\tilde{\phi}\tilde{\epsilon} + 2\tilde{\epsilon} + 2\tilde{\phi} + \tilde{A} \\ &\qquad - 8i\rho\left(z(1-z)\frac{d}{dz} + \tilde{\epsilon}(1-z) - \phi z + \frac{\tilde{M}-1}{4}(2z-1)\right) \end{aligned} \quad (21)$$

and where we have neglected the singular terms of the form

$$\frac{1-z}{z}2\epsilon(2\epsilon - 1) + \frac{z}{1-z}2\phi(2\phi - 1) \quad (22)$$

in H_{11} (and a similar terms with $\epsilon \to \tilde{\epsilon}$, $\phi \to \tilde{\phi}$ in H_{22}) since we assume from now on

$$\epsilon(2\epsilon - 1) = \phi(2\phi - 1) = \tilde{\epsilon}(2\tilde{\epsilon} - 1) = \tilde{\phi}(2\tilde{\phi} - 1) = 0. \quad (23)$$

The different choices for H_{12} proposed in equation (19) now appear to be natural since they will automatically lead to a polynomial expressions in z when the choice of the parameters $\epsilon, \tilde{\epsilon}, \phi, \tilde{\phi}$ is done according to equation (23). In the following, we will analyze in detail the case $H_{12} = C \sin x \cos x$. The algebraization corresponding to the three other cases can be done similarly.

In this case, the possible values for the parameters $\epsilon, \tilde{\epsilon}, \phi, \tilde{\phi}$ allow for four algebraization, for the wavefunction $\psi = (\psi_1, \psi_2)$, namely

$$\begin{aligned} &\text{type (i)}: \quad \psi = (p_n, \sin x \cos x q_{n-1}) \\ &\text{type (ii)}: \quad \psi = (p_{n-1} \sin x \cos x, q_n) \\ &\text{type (iii)}: \quad \psi = (p_n \sin x, q_n \cos x) \\ &\text{type (iv)}: \quad \psi = (p_n \cos x, q_n \sin x), \end{aligned}$$

where p_n, q_n, etc denote polynomials of degree n in the variable z.

Acting on an eigenfunction of type (i), the conditions for algebraic solutions are $\epsilon = \phi = 0, \tilde{\epsilon} = \tilde{\phi} = 1/2$. The operator $\tilde{H}$ can then be decomposed according to the prescription of section 2 leading to

$$\tilde{H} = \tilde{H}_1 + \tilde{H}_0 + \tilde{H}_{-1}, \tag{24}$$

with

$$\tilde{H}_1 = \begin{pmatrix} 8\mathrm{i}\rho\left(z^2 \frac{\mathrm{d}}{\mathrm{d}z} - \left(\frac{M-1}{2}\right) z\right) & -Cz^2 \\ \tilde{C} & 8\mathrm{i}\rho\left(z^2 \frac{\mathrm{d}}{\mathrm{d}z} - \left(\frac{\tilde{M}-3}{2}\right) z\right) \end{pmatrix} \tag{25}$$

$$\tilde{H}_0 = \left(4z^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2} + (4 - 8\mathrm{i}\rho) z \frac{\mathrm{d}}{\mathrm{d}z} + \rho^2\right) \mathbb{1}_2 + \begin{pmatrix} A' & Cz \\ 0 & 8z\frac{\mathrm{d}}{\mathrm{d}z} + \tilde{A}' \end{pmatrix} \tag{26}$$

and

$$\tilde{H}_{-1} = \begin{pmatrix} -4z\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2\frac{\mathrm{d}}{\mathrm{d}z} & 0 \\ 0 & -4z\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 6\frac{\mathrm{d}}{\mathrm{d}z} \end{pmatrix}, \tag{27}$$

with

$$A' = A - M^2 + 2\mathrm{i}\rho(M-1), \qquad \tilde{A}' = \tilde{A} + 4 - \tilde{M}^2 + 2\mathrm{i}\rho(\tilde{M} - 3). \tag{28}$$

Using the parametrization corresponding to type (i) for the wavefunction, we can easily obtain the form of the matrices $M_1, \tilde{M}_1, M_0$ and the conditions on the parameters leading to QES operators. In the present case, we get

$$M + \tilde{M} = 4n, \qquad (1 - 4n^2) + M\tilde{M} = \frac{C\tilde{C}}{16\rho^2}. \tag{29}$$

The condition involving M_0 fixes the difference between the constants $A, \tilde{A}$, namely

$$A - \tilde{A} = M^2 - \tilde{M}^2. \tag{30}$$

Considering the parametrization corresponding to type (ii) for the wavefunction, one obtains easily $\epsilon = \phi = 1/2, \tilde{\epsilon} = \tilde{\phi} = 0$ and the corresponding operator $\tilde{H}$ can be computed. The action of $\tilde{H}$ on an eigenfunction of the type (ii) leads, after some algebraic manipulations, to the same QES conditions as in the previous case, namely equations (29), (30).

The wavefunctions of the types (iii) and (iv) correspond respectively to $\epsilon = \tilde{\phi} = 0, \tilde{\epsilon} = \phi = 1/2$ and $\epsilon = \tilde{\phi} = 1/2, \tilde{\epsilon} = \phi = 0$. After some algebraic manipulations, we find the corresponding operators $\tilde{H}$ together with the matrices $M_1, \tilde{M}_1, M_0$. For these two types (iii) and (iv) the QES conditions read

$$M + \tilde{M} = 4n + 2, \qquad M\tilde{M} - 4n(n+1) = \frac{C\tilde{C}}{16\rho^2}, \tag{31}$$

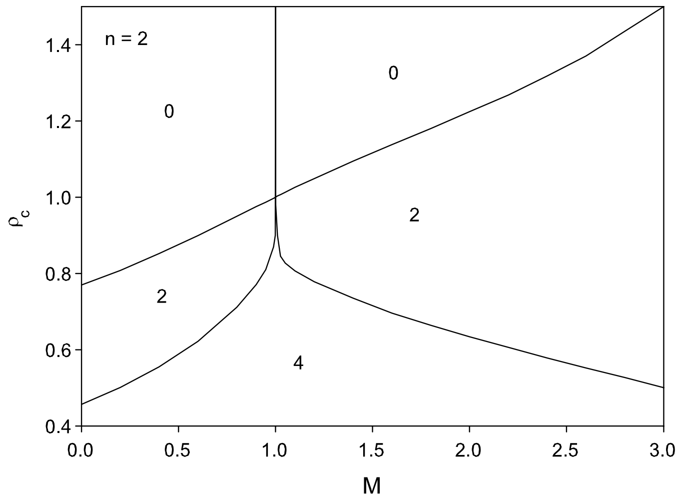

Figure 1. The critical value of ρ as a function of the coupling constant M for the type (i) solution and $n = 2$. The integers label the number of real algebraic eigenvalues.

while the condition involving M_0 leads, again, to (30). A consequence of these results is that the family of Hamiltonians (18) admits a double algebraization. The solutions of the types (i) and (ii) are available if conditions (29), (30) are fulfilled; solutions of the types (iii) and (iv) exist if conditions (31), (30) hold.

Example. In order to illustrate the results presented in this section, we studied the algebraic eigenvalues of the operator (18) for the solution of the type (i) and for $n = 1, 2$. The invariant vector space possesses $2n$ dimensions since the condition (5) imposes a constraint on the polynomials p_n, q_{n-1}. In the case $n = 1$, we find

$$E = \rho^2 + 2 \pm \sqrt{1 - \rho^2(1 + M)^2},$$

resulting in two real eigenvalues for $|\rho| < 1/|1 + M|$.

For $n = 2$, the four algebraic eigenvalues can be computed, in principle, but they take a particularly simple form in the cases $M = 1$ and $M = 3$:

$$\begin{aligned} &M = 1, \qquad E = 4 + \rho^2 (2 \text{ times}), \qquad && E = 8 + \rho^2 \pm 8\sqrt{1 - \rho^2} \\ &M = 3, \qquad E = 10 + \rho^2 \pm \sqrt{9 - 4\rho^2}, \qquad && E = 2 + \rho^2 \pm 2\sqrt{1 - 4\rho^2}. \end{aligned} \tag{32}$$

Solving the equation for generic values of M, ρ, we observed that the plane M, ρ is partitioned into regions admitting 4, 2 or 0 real, algebraic eigenvalues. This is illustrated on figure 1 where the critical values ρ_c are represented as functions of the coupling constant M. We observe that one of the critical value ρ_c becomes infinite in the limit $M \to 1$, in agreement with (32).

3.2. Hyperbolic case

The construction of the previous section can also be realized for the case where the trigonometric functions entering into the potentials are replaced by their elliptic counterpart.

The discussion of the different algebraizations turn out to be the same. Here, however, we will study in detail the algebraic properties of the operator given by

$$H = \begin{pmatrix} -\frac{d^2}{dx^2} - (\rho\cosh 2x - iM)^2 & C(\cosh 2x - 1) + \tilde{C} \\ D(\cosh 2x - 1) + \tilde{D} & -\frac{d^2}{dx^2} - (\rho\cosh 2x - i\tilde{M})^2 \end{pmatrix}, \tag{33}$$

where ρ is a free real parameter. One can assume $\tilde{M} > M$ without loosing generality. The gauge transformation is performed as follows:

$$\begin{aligned} \tilde{H} &= \exp(-\theta\cosh 2x) H \exp(\theta\cosh 2x), \\ &= \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{pmatrix}, \end{aligned} \tag{34}$$

On further substituting $z = \cosh 2x - 1$ and fixing the constant θ by means of $\theta = \frac{i\rho}{2}$, the different components of $\tilde{H}$ read

$$\begin{aligned} \tilde{H}_{11} &= -4z(z+2)\frac{d^2}{dz^2} - 4(z+1)\frac{d}{dz} - 8i\rho z\frac{d}{dz} - \rho^2 - 4i\rho\left(z^2\frac{d}{dz} - \frac{M-1}{2}z\right) \\ &\quad + 2i\rho(M-1) + M^2, \\ \tilde{H}_{12} &= Cz + \tilde{C}, \\ \tilde{H}_{21} &= Dz + \tilde{D}, \\ \tilde{H}_{22} &= -4z(z+2)\frac{d^2}{dz^2} - 4(z+1)\frac{d}{dz} - 8i\rho z\frac{d}{dz} - \rho^2 - 4i\rho\left(z^2\frac{d}{dz} - \frac{\tilde{M}-1}{2}z\right) \\ &\quad + 2i\rho(\tilde{M}-1) + \tilde{M}^2. \end{aligned} \tag{35}$$

Decomposing now the operator $\tilde{H}$ according to equation (1), we obtain

$$\tilde{H}_1 = \begin{pmatrix} -4i\rho\left(z^2\frac{d}{dz} - Nz\right) & Cz \\ Dz & -4i\rho\left(z^2\frac{d}{dz} - \tilde{N}z\right) \end{pmatrix}, \tag{36}$$

where we posed $N = \frac{M-1}{2}$, $\tilde{N} = \frac{\tilde{M}-1}{2}$. The form of $\tilde{H}_0$ and $\tilde{H}_{-1}$ can be obtained easily. Note that $(\tilde{H}_1)_{12} = Cz$ and $(\tilde{H}_1)_{21} = Dz$, so that $\delta = \delta' = 1$ in this case. Referring to the above general case and with

$$\psi = \begin{pmatrix} \alpha_0 z^n \\ \beta_0 z^n \end{pmatrix} + \begin{pmatrix} \alpha_1 z^{n-1} \\ \beta_1 z^{n-1} \end{pmatrix} + \cdots, \tag{37}$$

we can write the vector $\tilde{H}\psi$ according to

$$\tilde{H}\psi = \mathrm{diag}(z^{n+1}, z^{n+1}) M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} + \left(\mathrm{diag}(z^n, z^n)\tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + \mathrm{diag}(z^n, z^n) M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}\right) + \cdots, \tag{38}$$

where

$$\begin{aligned} M_1 &= \begin{pmatrix} -4i\rho(n-N) & C \\ D & -4i\rho(n-\tilde{N}) \end{pmatrix}, \\ \tilde{M}_1 &= \begin{pmatrix} -4i\rho(n-1-N) & C \\ D & -4i\rho(n-1-\tilde{N}) \end{pmatrix}, \\ M_0 &= -(4n^2 + 8i\rho n + \rho^2)\mathbb{1} + \begin{pmatrix} 4i\rho N + (2N+1)^2 & \tilde{C} \\ \tilde{D} & 4i\rho\tilde{N} + (2\tilde{N}+1)^2 \end{pmatrix}. \end{aligned} \tag{39}$$

The three necessary conditions for the operator $\tilde{H}$ to have a finite-dimensional invariant vector space can then be obtained in a straightforward way, the final results read

$$N + \tilde{N} = 2n - 1, \qquad 16\rho^2(n-N)(n-\tilde{N}) + CD = 0, \qquad \frac{\beta_0}{\alpha_0} = \frac{4\mathrm{i}\rho(n-N)}{C}, \tag{40}$$

the equation involving the metric M_0 imposes in turn

$$\tilde{C}\beta_0^2 + 4(N-\tilde{N})(2n+\mathrm{i}\rho)\beta_0\alpha_0 - \tilde{D}\alpha_0^2 = 0. \tag{41}$$

As a result, assuming a choice of the integer n, we end up with a family of QES operators labeled by the parameters N, ρ, C/D and $\tilde{C}$.

Different choices of the non-diagonal interactions H_{12} and H_{21} can be performed which lead to similar conditions between the cosmological constants. We will discuss these possibilities in the framework of periodic potentials (formulated in terms of trigonometric functions) largely discussed in the following section.

4. PT-symmetric QES equation with polynomial potential

In this section, referring to the unidimensional operator studied in [11] we will construct a PT-symmetric QES matrix Hamiltonian of the form

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathbb{1}_2 + M_4(x), \tag{42}$$

where $M_4(x)$ is the (2×2)-PT-symmetric matrix. The above Hamiltonian can be written in terms of components and we choose the potentials of the form

$$\begin{aligned} H_{11} &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^4 + \mathrm{i}Ax^3 + Bx^2 + \mathrm{i}Cx + D,\\ H_{12} &= \omega,\\ H_{21} &= \tilde{\omega},\\ H_{22} &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^4 + \mathrm{i}\tilde{A}x^3 + \tilde{B}x^2 + \mathrm{i}\tilde{C}x + \tilde{D}. \end{aligned} \tag{43}$$

In order to reveal the QES property, it is convenient to perform a gauge transformation according to

$$\tilde{H} = \exp(-\alpha x^3 - \beta x^2 - \gamma x) H \exp(\alpha x^3 + \beta x^2 + \gamma x). \tag{44}$$

The gauged Hamiltonian then simplifies considerably if

$$\alpha = -\frac{\mathrm{i}}{3}, \qquad \beta = -\frac{A}{4}, \qquad \gamma = \frac{\mathrm{i}}{2}\left(B - \frac{A^2}{4}\right), \qquad A = \tilde{A}, \qquad B = \tilde{B} \tag{45}$$

leading to the following expression:

$$\begin{aligned} \tilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 4\beta x\frac{\mathrm{d}}{\mathrm{d}x} - 2\gamma\frac{\mathrm{d}}{\mathrm{d}x} - 6\alpha\left[\left(x^2\frac{\mathrm{d}}{\mathrm{d}x} - mx\right) + \theta x\sigma_3\right]\\ + (-2\beta - \gamma^2) + \mathrm{diag}(D,\tilde{D}) + \omega\sigma_+ + \tilde{\omega}\sigma_-, \end{aligned} \tag{46}$$

where the constants C, $\tilde{C}$ have been redefined according to $C = \mathrm{i}(6\alpha(m-\theta)+6\alpha+4\beta\gamma)$, $\tilde{C} = \mathrm{i}(6\alpha(m+\theta)+6\alpha+4\beta\gamma)$.

However, in this form, the occurrence of an invariant finite-dimensional vector space of functions is not yet manifested in the sense that the operator $\tilde{H}$ does not preserve the vector space $(P_{m-\theta}, P_{m+\theta})^t$. In order to reveal such a possibility we can apply the technique of the first section. Here, we will follow [7] and perform a supplementary transformation on the operator $\tilde{H}$ with the matrix $S = \begin{pmatrix} 1 & \lambda\frac{\partial}{\partial x} \\ 0 & 1 \end{pmatrix}$. After some calculations, we obtain finally the form

$$\begin{aligned}\hat{H} &= S^{-1}\tilde{H}S, \\ &= \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Ax\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\left(B - \frac{A^2}{4}\right)\frac{\mathrm{d}}{\mathrm{d}x} + D + \frac{1}{4}\left(B - \frac{A^2}{4}\right)^2\right] - \tilde{\omega}\lambda\frac{\mathrm{d}}{\mathrm{d}x}\sigma_3 \\ &\qquad + 2\mathrm{i}\,\mathrm{diag}(J_+(n-2), J_+(n)) + \mathrm{diag}\left(\frac{A}{2}, -\frac{A}{2}\right) + \tilde{\omega}\sigma_- - \tilde{\omega}\lambda^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_+ \end{aligned} \tag{47}$$

with $J_+(n) \equiv x^2\frac{\mathrm{d}}{\mathrm{d}x} - nx$. Here, we have set $m = n-1$, $\tilde{D} = -A + D$, $\theta = 1$ and fixed the arbitrary parameter λ entering in the gauge transformation by means of $\omega = -2\mathrm{i}\lambda n$.

The Hamiltonian $\hat{H}$ manifestly preserves the finite-dimensional space $(P_{n-2}, P_n)^t$. Note that $A, B, D, \tilde{\omega}$ are free real parameters, n is a non-negative integer and λ is a free complex parameter.

5. Recurrence relations

In this section, we will express the formulation of the QES solution in terms of recurrence relations to the case of the PT-symmetric matrix Hamiltonian. We will see that the eigenvalue equation $H\psi = E\psi$ leads to a system of four terms recurrence relations. The solutions ψ are of the form

$$\psi(x) = \exp\left(-\frac{\mathrm{i}x^3}{3} - \frac{Ax^2}{4} + \frac{\mathrm{i}}{2}\left(B - \frac{A^2}{4}\right)x\right)\begin{pmatrix}\sum_{k=0}^{\infty} P_k(E)x^k \\ \sum_{l=0}^{\infty} Q_l(E)x^l\end{pmatrix}. \tag{48}$$

To solve the equation $H\psi = E\psi$ is equivalent to solve the following equation:

$$\hat{H}\begin{pmatrix}\sum_{k=0}^{\infty} P_k(E)x^k \\ \sum_{l=0}^{\infty} Q_l(E)x^l\end{pmatrix} = E\begin{pmatrix}\sum_{k=0}^{\infty} P_k(E)x^k \\ \sum_{l=0}^{\infty} Q_l(E)x^l\end{pmatrix}. \tag{49}$$

Then the above equation can be transformed into a *fourth*-order recurrence relation. It reads

$$A_k\begin{pmatrix}P_k \\ Q_{k+2}\end{pmatrix} + B_k\begin{pmatrix}P_{k-1} \\ Q_{k+1}\end{pmatrix} + C_k\begin{pmatrix}P_{k-2} \\ Q_k\end{pmatrix} + D_k\begin{pmatrix}P_{k-3} \\ Q_{k-1}\end{pmatrix} = 0, \tag{50}$$

where

$$\begin{aligned} A_k &= \begin{pmatrix} k(k-1) & 0 \\ -\tilde{\omega} & (k+2)(k+1)\end{pmatrix}, \\ B_k &= \begin{pmatrix} \left[\mathrm{i}\left(B - \frac{A^2}{4}\right) + \lambda\tilde{\omega}\right](k-1) & 0 \\ 0 & \left[-\lambda\tilde{\omega} + \mathrm{i}\left(B - \frac{A^2}{4}\right)\right](k+1)\end{pmatrix}, \\ C_k &= \begin{pmatrix} -D - \frac{1}{4}\left(B - \frac{A^2}{4}\right)^2 - A(k-2) - \frac{A}{2} + E & \tilde{\omega}\lambda^2 k(k-1) \\ 0 & -D - \frac{1}{4}\left(B - \frac{A^2}{4}\right)^2 - Ak + \frac{A}{2} + E\end{pmatrix} \\ D_k &= -2\mathrm{i}\begin{pmatrix}(k-n-1) & 0 \\ 0 & (k-n-1)\end{pmatrix}. \end{aligned} \tag{51}$$

In the present case, the recurrence relations are of fourth order, contrasting with other cases studied in the literature [12, 15] where they are of third order. Setting $\omega = \tilde{\omega} = 0$ the two recurrence relations decouple and the corresponding equations (e.g. the one for P_k) correspond to the scalar PT-invariant and QES quartic oscillator. It is also of fourth order; as a consequence, both P_0 and P_1 are arbitrary (P_0 fixes the normalization) and the other

$P_k, k \geqslant 2$ are determined recursively. The construction of the QES eigenvalues associated with this system is not as transparent as in the case of third-order recurrence where a common factor say P_n factorizes out of the P_k's, $k > n$. In the present case, the QES eigenvalues are obtained by solving the system

$$P_n(E, P_1) = 0, \qquad P_{n-1}(E, P_1) = 0, \tag{52}$$

which is linear in P_1. These conditions indeed lead to a truncation of the series for $\psi_1(x)$ defined in (48). Coming back to the full system (i.e. with $\omega \neq 0, \tilde{\omega} \neq 0$), it is easy to see that Q_0, Q_1, Q_2, Q_3 remain arbitrary (Q_0 set the normalization). The QES eigenvalues can be obtained by solving the system

$$\begin{aligned} &P_n(E, Q_1, Q_2, Q_3) = 0, \qquad &&P_{n-1}(E, Q_1, Q_2, Q_3) = 0, \\ &Q_{n+2}(E, Q_1, Q_2, Q_3) = 0, \qquad &&Q_{n+1}(E, Q_1, Q_2, Q_3) = 0, \end{aligned} \tag{53}$$

which turns out to be linear in Q_1, Q_2, Q_3.

6. Conclusions

In this paper, we have proposed a set of simple necessary and sufficient conditions for matrix-valued operators of a certain type to preserve a vector space of polynomials of fixed degrees. We have seen that the scalar Razhavi potential admits QES matrix extensions of several types. We also constructed a QES, matrix-valued PT-invariant Hamiltonian with polynomial potentials. Finally, by taking this last problem as an example, we have shown that the coupled differential equations can be transformed into a system coupled recurrence equations of fourth order.

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